Course Ordinary and Partial Differential Equations

Models, methods and applications

Contents

Part A. Ordinary Differential Equations

Overview

- What is an ordinary differential equation (ODE)?
- First-order, second-order and nth-order ODEs
- Linear and nonlinear ODEs
- ODEs with analytical solutions: integrating factor
- The Euler method for nonlinear ODEs

Motivational Example

- Bernoulli and Riccati equations
- ODEs in mathematical biology
- Radioactive decay
- Predator-prey models
- Interest-rate modelling

Existence Theory

- Sufficient conditions for existence and uniqueness in an interval
- Lipschitz continuity
- The Picard method of successive approximations
- The Gronwall inequality

Techniques

- Reducing second-order equations to a first-order system
- Polynomial-based solutions
- Homogeneous first-order equations
- First-order separable equations

Advanced ODEs and Applications

- *Stiff* ODEs
- Equations of motion and Lagrange's equations
- Lyapunov method
- Differential games

Systems of ODEs

- Vector and matrix differential equations
- The exponential matrix
- Cayley-Hamilton theorem
- Power series solutions

Part B. Two-Point Boundary Value Problems (BVP)

- Reduction to normal form
- Dirichlet, Neumann and Robin boundary conditions
- Time-independent convection-diffusion equation
- Convection-dominated problems
- Applications

Special Boundary Value Problems

- Sturm-Liouville problem and self-adjoint differential operators
- Bessel equation
- BVPs in conservative and non-conservative forms
- Green's function

Integral Equations

- Volterra and Fredholm equations of the first and second kinds
- Hilbert-Schmidt kernels
- Solving integral equations
- Applications of integral equations

Numerical Solution of ODEs

- Method of Lines (MOL)
- ODE Solver libraries
- Test 101: Numerical solution of the heat equation

Part C. Methods for Solving Differential Equations

Integral Transforms

- Converting differential equations to algebraic equations
- The convolution transform
- The Mellin transform
- Applications

The Laplace Transform

- One-sided and two-sided Laplace transforms
- Solving initial value problems with the Laplace transform
- Inverse Laplace transform
- Some properties of the Laplace transform

- Appications to probability
- Greens' function by Laplace Transform

Fourier Series
- Even and odd functions
- Fourier series of piecewise smooth functions
- Sine and cosine series
- Pointwise and uniform convergence of Fouries series
- Gibbs Phenomenon
- Fejer and Cesaro sums

Advanced Fourier Series
- The sinc function
- Fejer and Dirichlet kernels
- Cesaro sums
- Nascent delta functions
- Whittaker's cardinal function

The Fourier Transforms and Integrals
- Classical rules
- Fourier Integral theorem
- Fourier transform in the complex plane
- A short introduction to distributions
- The Dirac delta function

Classical Orthogonal Polynomials
- Orthogonality: recurrence relation
- Generating functions
- Differential equations satisfied by orthogonal polynomials
- Application areas

Part D. Elliptic PDEs

Categories
- Laplace, Poisson (diffusion)
- Convection-diffusion equations
- Conservative and non-conservative equation forms
- Diffusion-reaction equations

Boundary Value Problems for Laplace's Equation
- Rectangular, circular and spherical regions
- Rectangular, polar, cylindrical and spherical coordinates
- Spherical harmonics

Boundary Value Problems for Elliptic Equations
- Well-posed problems
- Dirichlet, Neumann and Robin boundary conditions
- Radiation boundary conditions
- Bounded, infinite and semi-infinite domains
- Fichera theory for degenerate elliptic equations

An Introduction to Free Boundary Problems
- Elliptic variational inequalities
- Existence and regularity
- Obstacle problem
- Filtration problem
- Other applications

Part E. Hyperbolic PDEs
- Second-Order Hyperbolic PDE
- Cauchy problem for hyperbolic systems
- Cauchy problem for the wave equation
- The Method of Characteristics
- Kirchoff's formula

Special Methods
- d'Alembert's formula
- Goursat problem
- Lorentz transformation
- Uniqueness of the mixed initial and boundary-value problem

Model Hyperbolic PDE: Vibrating String
- General solution
- Initial and boundary conditions
- Forced vibrations
- Fourier method for equation of vibrating string
- Vibrations of a membrane
- Circular drum and Bessel's function

First-Order Hyperbolic PDEs
- Categories
- Number of independent variables
- Scalar equations; systems of equations
- Linear, semilinear and quasilinear equations
- Initial value problems
- Nonexistence and nonuniqueness of solutions

Part F. Parabolic PDEs

Categories of Parabolic PDEs
- Diffusion Equations
- Diffusion-reaction
- Convection-diffusion
- Convection-diffusion-reaction
- Examples and application

Initial Boundary Value Problems (IBVP)
- Defining the space domain
- Cauchy problem on infinite and semi-infinite domains
- Bounded domain
- Domain truncation/domain transformation
- Boundary conditions (Dirichlet, Neumann, Robin, non-local)

Qualitative Properties of Parabolic PDEs
- Maximum principle
- Uniqueness of solution of IBVPs

- Estimates on solution growth
- Energy inequalities

Method of Separation of Variables
- Applicability and use
- Eigenvalues and eigenfunctions
- Example: one-dimensional heat and wave equations
- Schrödinger wave equation
- Biorthogonal series

An Introduction to Moving Boundary Problems
- Parabolic variational inequalities
- Introductory examples
- The Stefan problem
- Other applications

Part G. PDEs in Computational Finance
Overview and Main Use Cases
- One-factor and two-factor models
- Form of the PDE and boundary conditions
- Domains in which PDEs are defined
- Domain truncation and domain transformation
- Specific models (equity, fixed income, hybrid)

PDE Preprocessing and Transformation
- Mapping infinite and semi-infinite domains to bounded domains
- Tips, rationale and guidelines for domain transformation
- What does domain transformation do to a PDE?
- Advantages of domain transformation

Specific PDE Categories
- General convection-diffusion-reaction PDE on unbounded domains
- PDEs with and without mixed derivatives
- Self-adjoint form
- PDEs with and without convection term
- PDEs on bounded domains

Mathematical Foundations
- Cauchy problems and well-posedness
- Fichera theory and Feller condition
- Reduction to canonical form
- Convection-dominance
- Free boundary problems and early exercise feature

Test Cases
- Classic one-factor Black Scholes PDE
- Two-factor basket options
- Cox-Ingersoll-Ross (CIR)
- Heston and SABR model
- Convertible bonds
- Asian options
- One-factor and two-factor Hull-White models

Part H. Supported Results and Identities
Vector Analysis
- Vectors and scalars
- Scalar and vector fields
- Dot and cross product
- Gradient, divergence and curl
- Integration (ordinary, line, surface and volume)
- Computing normals to surfaces

Theorems and Identities
- Gauss divergence theorem
- Stokes and Kelvin-Stokes theorems
- Green's theorem
- Green's first and second identities

Application Areas
- Potential theory
- Electromagnetic theory
- Differential geometry
- Finite Element Method (FEM)

Software Libraries
- Boost C++ libraries
- Eigen C++ Libraries
- MathNet C#
- Python numpy and scipy
- Daniel Duffy's PDE solvers for option pricing
- F# libraries

Your Trainer
Daniel J. Duffy started the company Datasim in 1987 to promote C++ as a new object-oriented language for developing applications in the roles of developer, architect and requirements analyst to help clients design and analyse software systems for Computer Aided Design (CAD), process control and hardware-software systems, logistics, holography (optical technology) and computational finance. He used a combination of top-down functional decomposition and bottom-up object-oriented programming techniques to create stable and extendible applications (for a discussion, see Duffy 2004 where we have grouped applications into domain categories). Previous to Datasim he worked on engineering applications in oil and gas and semiconductor industries using a range of numerical methods (for example, the finite element method (FEM)) on mainframe and mini-computers.

Daniel Duffy has BA (Mod), MSc and PhD degrees in pure and applied mathematics and has been active in promoting partial differential equation (PDE) and finite difference methods (FDM) for applications in computational finance. He was responsible for the introduction of the Fractional Step (Soviet Splitting) method and the Alternating Direction Explicit (ADE) method in computational finance. He is also the originator of the exponential fitting method for time-dependent partial differential equations.

He is also the originator of two very popular C++ online courses (both C++98 and C++11/14) on www.quantnet.com in cooperation with Quantnet LLC and Baruch College (CUNY), NYC. He also trains developers and designers around the world. He can be contacted dduffy@datasim.nl for queries, information and course venues, in-company course and course dates