Distance Learning Course Pure Mathematics Foundations

Structure of Course and Student-Trainer Interaction

This course takes an incremental/inductive approach by first examining model problems in detail and then extending them to larger applications. Student progress is measured by working on graded exercises. Regular communication takes places by e-mail and Skype.

Students receive a certificate on successful completion of the course.

Preamble and Review: College/High-School Mathematics

This part reviews a number of basic mathematical techniques and methods as preparation for the main part of the course. The goal of this section is to (re)acquaint yourself with the notation of mathematics, learning mathematics by doing mathematics and becoming comfortable before the main topics are introduced.

You may skip this section if you already know these topics. In that case you can progress straight to Part A.

Algebra

- Linear and quadratic equations
- Simultaneous linear equations
- Inequalities
- Arithmetic and geometric progressions

More Algebra

- Mathematical induction
- The Binomial Theorem
- Polynomial equations and their roots
- Permutations and combinations
- Infinite sequences and series

Plane Trigonometry

- Angles and arc length
- Trigonometric functions of one and two angles
- Applications
- Fundamental relations and identities

Plane Analytic Geometry

- Coordinates and loci
- The straight line
- The circle and ellipse
- Transformation of coordinates
- Polar coordinates

Vector Analysis

- Vectors and scalars
- Dot and cross product
- Gradient, divergence and curl

The Difference Calculus

- The algebra of operators
- The different kinds of operators
- Application of the Difference Calculus
- Linear difference equations with constant coefficients

A System of Thinking on How to solve Mathematical Problems

General Guidelines

- Understand the problem
- Make a plan
- Carry out the plan
- Review your work

Heuristics for Mathematical Reasoning (Georg Polya)

- Analogy
- Generalisation
- Induction
- Variation of the Problem
- Specialisation
- Decomposition and Recombination
- Working backwards

Part A. Algebra and Number Systems

Sets

- Describing sets
- Equal sets; subsets of a set
- Set operations (union, intersection, difference, complement)
- Cartesian product of sets
- Finite and infinite sets

Mappings between Sets
- One-to-one (injective) mapping
- Onto (surjective) mapping
- Bijective mapping
- Composition of mappings

Relations and Operations
- Binary and n-ary relations
- Reflexive, symmetric and transitive relations
- Equivalence relations
- Partitions of a set
- Ordering in sets
- Isomorphisms and permutations
- Tuples

The Natural Numbers
- Peano postulates
- Mathematical induction
- Order relations
- Trichotomy Law

The Integers
- Constructing integers from the natural numbers
- Operations on integers
- Number systems (decimal, binary, hexadecimal)
- Prime numbers and composite numbers

Theorems and Properties
- Greatest Common Divisor
- Least Common Multiple
- Division algorithm
- Linear congruences
- Fundamental Theorem of Arithmetic

Rational Numbers
- Constructing the rational numbers
- Reduction to lowest terms
- Archimedean principle
- Decimal representation

Real Numbers
- Dedekind cuts
- Operations on real numbers
- Irrational and transcendental numbers
- Completeness of the real numbers

Complex Numbers
- What is a complex number?
- Operations on complex numbers
- Complex polynomial equations
- Trigonometric and hyperbolic functions
- Many-valued functions; principal value

An Introduction to Rings
- Types of rings
- Characteristic; divisors of zero
- Principal ideals
- Quotient and Euclidean rings

An Introduction to Fields and Integral Domains
- Unit, associate, divisor
- Unique factorization
- Division rings
- Fields

Polynomials
- Polynomial forms
- Division algorithm
- Remainder and Factor theorems
- Greatest common divisor
- Unique Factorisation Theorem

Matrix Polynomials
- Normal form of a λ matrix
- Polynomials with matrix coefficients
- Division algorithm
- Characteristic roots and vectors of a matrix
- Conic and quadric surfaces

An Introduction to Group Theory
- Simple properties of groups
- Subgroups
- Cyclic and permutation groups
- Isomorphisms
- Quotient groups

Part B. Real Analysis Fundamentals
Continuous Functions
- Intuitive concept of continuity
- Precise definition of limit of a function
- Basic limit theorems
- Squeezing principle

Special Kinds of Functions
- Piecewise continuous functions
- Discontinuous functions
- Monotonic functions
- Convex and concave functions

Differential Calculus
- Motivation: velocity of a projectile
- Definition of derivative
- Examples
- The algebra of derivatives

Advanced Differentiation

- The derivative as a slope
- Chain rule of differentiation for composite functions
- Implicit differentiation
- Numerical differentiation

Theorems

- Mean-value theorem
- Cauchy's mean-value formula
- Finding maxima and minima of functions
- Rolle's theorem
- Taylor's theorem

Integration

- Relation between integration and differentiation
- First and second fundamental theorems of calculus
- Integration by parts
- The natural logarithm and exponential functions

Part C. Sequences and Series

Numerical Sequences

- Real and complex sequences
- Convergent and divergent sequences
- Cauchy sequences
- Subsequences
- Monotonic sequences

Numerical Series

- From sequences to series
- Convergent series: necessary and sufficient conditions
- Telescoping series
- Series of nonnegative terms
- Alternating series
- Geometric series

Tests for Convergence

- Root test and ratio test
- Integral test
- Conditional and absolute convergence
- Cauchy test
- Abel test
- Weierstrass M-test

Sequence of Functions

- Limit of a sequence of functions
- Pointwise convergence
- Discrete sequences of functions
- Uniform convergence

Series of Functions

- Power series
- M-test for functions
- The Taylor's series
- The binomial series

Part D. Functions of Several Variables

Overview

- Mappings in Euclidean spaces
- Scalar, vector and vector-valued functions
- Composite functions
- Application areas

Partial Differentiation

- Differentiable functions
- Higher derivatives
- Differentiation of implicit functions
- Differentials and directional derivatives
- Taylor's theorem

Theorems

- Inverse function theorem
- Implicit function theorem
- Rank theorem

Advanced Topics

- Gradient and Jacobian
- Hessian
- Inverse of a transformation
- Chain rule for differentiation

Part E. Integration and Function Spaces

Riemann Integration

- Riemann integral as a limit of a sum
- Differentiation and integration
- Integrals of sequences and series
- Improper Riemann integrals
- Nonintegrable functions

Riemann-Stieltjes Integral

- Formulation
- Integration of vector-valued functions
- Functions of bounded variation
- First and second mean value theorems
- Riesz Representation Theorem

Measure Theory "101" Introduction

- Geometric motivation of measure
- Measurable and non-measurable sets
- Lebesgue measure
- Almost everywhere
- Borel sets
- Measurable Functions

The Lebesgue Integral for Bounded Functions

- Geometric interpretation
- Theorems
- Bounded convergence theorem
- Relationship between Riemann and Lebesgue integrals

The Lebesgue Integral for Unbounded Functions

- Motivation
- Lebesgue's Dominated Convergence Theorerm
- Fatou's lemma
- Monotone convergence theorem
- Approximating integrable functions by continuous functions

Double Lebesgue Integrals

- Lebesgue measure in the plane
- Fubini's theorem
- Fubini-Tonelli-Hobson theorem
- Multiple Lebesgue integrals

Part F. Introductory Functional Analysis

Overview

- Short history of functional analysis
- Types of spaces
- Linear and nonlinear transformations between spaces
- Infinite-dimensional and finite-dimensional spaces
- Application areas

Metric Spaces

- Distance (metric) function
- Cauchy-Schwarz inequality
- Discrete metric spaces
- Isometry

Topological Considerations

- Open and closed sets
- Continuity
- Homeomorphic metric spaces
- Topological spaces
- Convergence and completeness
- Compactness

Applications of Functional Analysis

- Hilbert and Banach Spaces
- Orthogonal polynomials
- Fixed point analysis; Banach Contraction Mapping
- Matrices and matrix norms

Normed Linear Spaces (NLS)

- What is a norm?
- Hölder's and Minkowski's inequalities
- Examples of NLS
- Linear transformations
- Isomorphisms
- Finite-dimensional spaces

Inner Product Spaces

- What is an inner product?
- Schwarz inequality
- Orthogonality
- (Modified) Gram-Schmidt orthogonalisation process
- Orthogonal bases

Part G. Discrete Mathematics

- Complexity Analysis
- Notation
- Best-case complexity
- Worst-case complexity
- Average complexity
- Big O and small o notation

Introduction to Graph Theory

- What is a graph?
- Graph and digraph
- Weighted digraph
- Paths and connectivity
- Graph data structures

Graph Algorithms

- Operations on graphs
- Depth-first and breadth-first search
- Shortest paths
- Spanning trees
- Connected components

H. *Introduction* to Probability and Stochastic Analysis

Review of Probability

- Sample space and events
- Axioms of probability
- Conditional probability
- Independent events

Random Variables

- Distribution functions
- Discrete and continuous random variables
- Probability mass and probability density functions
- Multiple random variables

Mathematical Foundations of Markov Chains

- Stochastic matrices (left, right, double)
- Substochastic matrices
- Nonnegative and positive matrices
- Perron-Frobenius theorem
- Spectral analysis of stochastic matrices

An Introduction to Markov Chains

- The Markov property
- State space, stochastic matrices and state diagrams
- Continuous and discrete time Markov chains
- Transition probability and transition rate (intensity) matrix
- Kolmogorov forward equation
- Steady-state analysis and limiting distributions
- Hitting times

Applications of Markov Chains

- Random walk
- Markov Chain Monte Carlo (MCMC)
- Multidimensional integrals
- Bayesian statistics

An Introduction to Bayesian Statistics

- Prior and posterior distributions
- Bayesian versus frequentist approach
- Likelihood function
- Bayesian inference

An Introduction to Stochastic Differential Equations (SDEs)

- From ordinary differential equations (ODEs) to SDEs
- One-factor and n-factor SDEs
- The Ito and Stratonovich integrals
- SDE: existence and uniqueness results
- Weak and strong solutions

Examples and Applications of SDEs

- Geometric Brownian Motion (GBM)
- Mean-reverting SDEs (Ornstein–Uhlenbeck)
- Turbulent diffusion
- Hydrology

Numerical Solution of SDEs

- Analytical solution
- Euler and Milstein methods
- Drift-adjusted predictor-corrector method
- Random number generators

Your Trainer

Daniel J. Duffy started the company Datasim in 1987 to promote C++ as a new object-oriented language for developing applications in the roles of developer, architect and requirements analyst to help clients design and analyse software systems for Computer Aided Design (CAD), process control and hardware-software systems, logistics, holography (optical technology) and computational finance. He used a combination of top-down functional decomposition and bottom-up object-oriented programming techniques to create stable and extendible applications (for a discussion, see Duffy 2004 where we have grouped applications into domain categories). Previous to Datasim he worked on engineering applications in oil and gas and semiconductor industries using a range of numerical methods (for example, the finite element method (FEM)) on mainframe and mini-computers.

Daniel Duffy has BA (Mod), MSc and PhD degrees in pure and applied mathematics and has been active in promoting partial differential equation (PDE) and finite difference methods (FDM) for applications in computational finance. He was responsible for the introduction of the Fractional Step (Soviet Splitting) method and the Alternating Direction Explicit (ADE) method in computational finance. He is also the originator of the exponential fitting method for time-dependent partial differential equations.

He is also the originator of two very popular C++ online courses (both C++98 and C++11/14) on www.quantnet.com in cooperation with Quantnet LLC and Baruch College (CUNY), NYC. He also trains developers and designers around the world. He can be contacted dduffy@datasim.nl for queries, information and course venues, in-company course and course dates