The Finite Difference Method (FDM) for Ordinary, Partial and Stochastic Differential Equations

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Summary of Contents

The course consists of eight modules. Each module deals with a single major area and it uses the results from its predecessor module. Its output is then needed by its successor module. In this way we reduce the amount of coupling and thus help flatten the learning curve.

Part A Mathematical Foundations and Background

Introduces the mathematical and numerical analysis concepts that are needed in order to understand the finite difference method and its application to computational finance, in particular continuous and discrete mathematics fundamentals. The presented material can also be used as a standalone reference.

Continuous Functions

- Intuitive concept of continuity
- Precise definition of limit of a function
- Basic limit theorems
- Squeezing principle

Special Kinds of Functions

- Piecewise continuous functions
- Discontinuous functions
- Monotonic functions
- Convex and concave functions

Differential Calculus

- Motivation: velocity of a projectile
- Definition of derivative
- Examples
- The algebra of derivatives

Advanced Differentiation

- The derivative as a slope
- Chain rule of differentiation for composite functions
- Implicit differentiation
- Numerical differentiation

Part B Finite Difference Method for Ordinary Differential Equations (ODEs)

This module discusses ODEs from three main perspectives; first, we introduce first order ODEs and we discuss their qualitative properties such as existence and uniqueness of linear and nonlinear scalar ODEs and ODE systems. Second, we introduce the Finite Difference Method by applying it to ODEs. Finally, we discuss the advantages of this approach in the context of this course, for example:

- Building a common mathematical and numerical notation and terminology that permeates all aspects of the course.
- Introducing specific finite difference schemes (by their name) and using them to approximate ODEs. The very same schemes will be extended and applied to PDEs and SDEs in later modules.
- Gaining insights into FDM by examining concrete examples in detail (hands-on approach) before moving to larger problems.

Ordinary Differential Equations: Theory

Overview

- What is an ordinary differential equation (ODE)?
- First-order, second-order and nth-order ODEs
- Linear and nonlinear ODEs
- ODEs with analytical solutions: integrating factor
- The Euler method for nonlinear ODEs

Motivational Examples

- Bernoulli and Riccati equations
- ODEs in mathematical biology
- Radioactive decay
- Predator-prey models
- Interest-rate modelling

Existence Theory

- Sufficient conditions for existence and uniqueness in an interval
- Lipschitz continuity
- The Picard method of successive approximations
- The Gronwall inequality

Ordinary Differential Equations: Numerics

Theoretical Foundations

- Single-step and multi-step methods
- Explicit and implicit schemes
- Stability and accuracy
- Discrete maximum principle

Popular Schemes

- Explicit and implicit Euler methods
- Crank-Nicolson method
- Predictor-Corrector methods
- Methods for systems of ODEs
- Examples

Advanced Topics

- Exponential Fitting
- Extrapolation and improving accuracy
- Matrix differential equations
- Stiff ODEs

Part C Foundations of the Finite Difference Method

We introduce the mathematical background to the finite difference method for initial boundary value problems for parabolic partial differential equations. This module encapsulates in one place all the background information that is needed to construct stable and accurate finite difference schemes for time-dependent problems. The schemes will be applied to one-factor and two-factor finance PDEs in later chapters. The advantage is that the sections discuss finite difference schemes for generic PDEs which will then be applied to finance PDEs in finance.

Theoretical Underpinnings

- Fourier analysis of PDEs
- Fourier transform and inverse Fourier transform
- Fourier Transform for the advection and diffusion equations

Discrete Fourier Transform (DFT)

- Finite and infinite dimensional sequences
- Using DFT for finite difference schemes
- von Neumann stability (amplification factor/symbol)
- Test Case A-Z: the advection (convection) equation

Convergence Analysis

- Consistency and stability
- Unconditional and conditional stability
- First-order and second-order accuracy
- Test Case: one-dimensional heat equation

Boundary Value Problems: Continuous Case

- Time-independent convection-diffusion equations
- Conservative and non-conservative equation forms
- Boundary conditions: Dirichlet, Neumann, Robin
- Existence and uniqueness results

Boundary Value Problems: Discrete Approximation

- Approximating derivatives by divided differences
- Approximating boundary conditions
- Accuracy and stability requirements
- Assembling the discrete matrix system

Numerical Linear Algebra

- Types of matrices (e.g. M-matrices)
- Solving matrix systems by direct methods
- LU decomposition
- Thomas algorithm for tridiagonal systems

Part D Finite Difference Method for One-Factor PDEs

This module introduces time-dependent partial differential equations in one space variable and their numerical approximation by a variety of finite difference schemes. In general, we can view a pde as a composition of a first-order ODE in time and a second-order ODE in space. In order to have a unique solution we must prescribe an initial condition and auxiliary boundary conditions.

Summarising, this module discusses how to specify initial boundary value problems for time-dependent PDEs and how to approximate them by popular finite difference schemes. We note that the schemes we use for one-factor problems will be used and generalised to two-factor problems as we shall in Part G.

PDE Preprocessing

- Domain truncation and domain transformation
- Change of variables
- PDEs in conservative and non-conservative form
- The Fichera theory for boundary conditions

Discretisation Strategies

- Semi-discretisation and Method of Lines (MOL)
- Simultaneous (full) discretisation in time and space
- Explicit and implicit methods
- First-order and second-order accuracy

Some popular Schemes

- Fully implicit and Crank-Nicolson
- Alternating Direction Explicit (ADE)
- Upwinding and exponential fitting for convection-dominated problems
- Schemes for PDEs in conservative form

Part E Stochastic Differential Equations (SDEs)

In general terms, an sde has more or less the same form as an ode but with a random term added on. SDEs are important equations when we model random processes in finance, fluid dynamics and simulation. It is important to gain an understanding of SDEs because together with ODEs and PDEs they are the foundation for numerical simulation in many application areas.

The main topics are:

- An introduction to stochastic differential equations (SDEs) and their numerical approximation.
- Some popular finite difference schemes for SDEs and applications to Geometric Brownian Motion (GBM).
- Option pricing using Monte Carlo simulation and C++ code for one-factor SDE.
- Fundamental results and theorems; Feyman-Kac formula, Kolmogorov backward and forward (Fokker-Planck) equations and their applications.

Stochastics Background

- Stochastic Differential Equations (SDE)
- The Ito formula
- The equivalence between SDEs and PDEs
- Where are SDEs used?

Monte Carlo Simulation

- Discretising SDEs
- Path simulation
- Random number generation
- Option pricing

Numerical Approximations of SDEs

- Euler-Maruyama
- Milstein
- Predictor-Corrector
- Drift-adjusted Predictor-Corrector

Part F Some Test Cases and Applications

We introduce a number of modern and popular finite difference methods to approximate the solution of initial boundary value problems for one factor differential equations. In particular, we apply the schemes from Part D to real life applications. To our knowledge, this is the only course that discusses these methods as well as their comparative strengths together with their applications to option pricing and hedging.

We also devote two sections to Sensitivity Analysis and we propose at least five methods to compute the derivatives of solutions of initial boundary value problems with respect to underlying parameters. In finance, these are sometimes called option greeks.

Computing Sensitivities and Greeks

- What are sensitivities and why do we need them?
- Divided differences ("bumping")
- Sensitivities as solutions of PDEs (Continuous Sensitivity Equation (CSE))
- Complex Step Method (CSM)
- Using Cubic Splines

Test Case: Black Scholes PDE A-Z

- Initial boundary value problem for Black Scholes PDE
- Choosing an approximation strategy
- Comparing with analytical solution
- Early exercise features
- Computing the greeks
- Examples in finance: delta, gamma, vega

Part G Advanced Finite Difference Method

In this section we discuss several popular finite difference methods to approximate the solutions of the PDEs describing two-factor option pricing. We discuss the Alternating Direction Implicit (ADI) method and the method of Fractional Steps ("Soviet Splitting") that originated in the United States and the former Soviet Union in the 1960's, respectively. We apply them to several PDEs in computational finance. Of particular importance is the problem of approximating the mixed derivatives in the PDE to ensure that the resulting scheme is monotone and hence does not lead to spurious oscillations.

We also discuss MOL and ADE for linear and nonlinear PDEs and we compare them with ADI and splitting methods.

The Method of Lines (MOL) Overview

- Semi-discretisation
- Vertical MOL and horizontal MOL (Rothe's method)
- Example: one-dimensional heat equation
- Advantages of MOL
- Application areas

MOL in Detail

- Stiff and non-stiff ODEs
- Linear and nonlinear systems
- Incorporating non-Dirichlet boundary conditions into MOL
- Adaptive and non-adaptive ODE solvers

MOL PDE Examples

- Black Scholes
- Cox Ingersoll Ross (CIR)
- Uncertain Volatility Model (UVM)
- CEV model
- Pde for credit value adjustment (CVA)
- MOL using Boost C++ odeint

ADE for one-Factor and Two-Factor Problems

- Background and motivation
- Saul'yev, Barakat-Clark and Larkin variants
- ADE for convection terms
- Conditional consistency; stability
- Boundary conditions

Two-Factor Contenders

- Alternating Direction Implicit (ADI)
- Splitting (Fractional Steps method)
- ADE in two dimensions
- Other methods

The ADI Method

- Using ADI for two-factor PDE
- Mixed derivatives using Craig-Sneyd
- Test cases: basket options and Heston model
- Generalising the ADI method

The Operator Splitting Method

- Yanenko, Marchuk and Strang splittings
- Explicit and implicit splitting
- Handling mixed derivatives and boundary conditions
- Splitting and predictor-corrector methods
- Marchuk 1-2-2-1 model

The ADE Method

- Origins and background; how it differs from ADI and splitting
- Motivating ADE: from heat pde to convection-diffusion and mixed derivatives
- One-sided and centred variants of ADE
- ADE in 3 factors

Comparing ADI, Splitting, MOL and ADE Methods

- How they handle mixed derivatives
- Boundary conditions
- Accuracy and robustness of the schemes
- Improving accuracy
- Can the scheme be parallelised?

Mixed Derivatives

- Removal using PDE canonical form
- Modelling correlation: extreme cases
- Craig-Sneyd, Yanenko
- Stress-testing mixed derivatives
- Test case: compare ADI, splitting and ADE for Heston model

Test Cases

- Basket options
- Heston model
- Asian options
- Anchoring model
- Analytical solutions

Part H The FDM "Process" – How to do it.

The goal of this module is to produce accurate and robust approximations to the solution of differential equations using the techniques from the first seven modules. We take a defined and reproducible process-driven approach by taking a PDE model, approximating this model by one or more finite difference schemes and then implementing the resulting algorithms in a favourite language such as C++ or Python. The best way to learn all this stuff is to take a concrete project and work it out A-Z.

The main learning objectives and skills achieved in this module are:

- Learn the full mathematical/numerical/programming trajectory, from problem description to number crunching.
- Develop skills to analyse and understand the trade-offs and alternative approaches at each stage of the trajectory.
- An extensive collection of modern and industrial-strength finite difference schemes that you can apply to a range of ODEs, PDEs and SDEs.

Gaining Insight (Heuristics and How to Solve it)

- Analogy
- Variation of the problem
- Auxiliary problem
- Precedence
- Decomposition and recombination
- Generalisation, specialisation and induction

Basic Accuracy Testing (Proof-of-Concept)

- Estimating the local truncation error
- The effects of domain truncation and domain transformation
- Which boundary conditions are optimal?
- Choice of matrix solver

Improving Accuracy and Robustness

- Non-smooth payoffs
- Convection-dominance and exponential fitting
- Computing option sensitivities
- Automatic testing: producing a report

Testing Accuracy: Sparring Partners

- Is there an analytic/quasi-analytic solution to test against?
- Monte Carlo solution
- Binomial method solution
- Two-factor analytic solution